Switch and routers architectures: matching algorithms

Matching algorithms

Given a bipartite graph with weighted edges:

\[ w(mWM) \geq \frac{1}{2} w(MWM) \]

where:

- \( w(mWM) \) is the weight of the maximal weighted match: with this match a further edge can not be added;

- \( w(MWM) \) is the weight of the maximum weighted match: this match maximizes the weight.

In practise:

\[ MWM \Rightarrow mWM \]

but the viceversa does not hold. To compute a \( mWM \):

- compute analytically iLQF;

- compute a greedy \( mWM \) called \( GWM \).

The Greedy Weighted Match

Heuristics can be:

- sort \( N^2 \) edges: the final complexity is \( O(N^2 \log N) \);

- look at inputs and sort edges based on their weight (possible parallel implementation): the final complexity is \( N' \cdot O(N \log N) \).

Proof

Assume \( \bullet \) an edge of the \( GWM \) and \( \bullet \) be an edge of the \( MWM \). In a graphical way:

\[ \bullet \]

\[ \bullet \]

\[ \bullet \]

\[ \bullet \]

\[ \bullet \]

\[ \bullet \]

\[ \bullet \]

* \( N \) is the number of ports
Let:

\[ G = \{\text{edges} \in E : \mathcal{GWM}\} \]
\[ M = \{\text{edges} \in E : \mathcal{MWM}\} \]

where \( E \) is the edge set. Let:

\[ M_1, M_2, M_3 = 0 \]
\[ G_1, G_2, G_3 = 0 \]

be sets corresponding to different situations. Let be \( e_G \) be the edge with the highest weight at each iteration, and \( e_M \) the edge corresponding to the actual choice of \( \mathcal{MWM} \). Then the following possibilities are there:

1. \( e_G = e_M \): in this case remove both edges from \( G \) and \( M \) and put into sets \( G_1 \) and \( M_1 \):

\[ M_1 = M_1 \cup \{e_M\} \]
\[ G_1 = G_1 \cup \{e_G\} \]

this situation happens when:

2. when, instead there are those situations:

\[ e_G \\ e_M \\ e_G \\ e_M \]

it means that:

\[ w(e_G) \geq w(e_M) \]

In this case, the \( \mathcal{GWM} \) selects \( e_G \) instead of \( e_M \) although this is not a good decision, but it is local optimal. Then:

\[ M_2 = M_2 \cup \{e_M\} \]
\[ G_2 = G_2 \cup \{e_G\} \]

3. the last possibility happens when:

\[ e_G \\ e_M \\ e_G \\ e_M \]
in this case:

\[
\begin{align*}
    &w(e_G) \leq w(e_M) \\
    &w(e_G) \leq w(e_M') \\
    &0.252w(e_G) \leq w(e_M') + w(e_M'')
\end{align*}
\]

Therefore:

\[
    w(e_G) \leq \frac{1}{2} \left[ w(e_M') + w(e_M'') \right]
\]

And:

\[
    M_3 = M_3 \cup \{ e_M'' \} \cup \{ e_M' \} \\
    G_3 = G_3 \cup \{ e_G \}
\]

These 3 cases are exhaustive if:

\[
    G \rightarrow |\mathcal{GWM}| = \mathcal{N} \\
    M \rightarrow |\mathcal{MWM}| = \mathcal{N}
\]

this implies that the \(\mathcal{MWM}\) should also consider edges with null weight.

**Example 1 (Matching admissibility)** Consider:

it can not be accepted because is missing this edge:

Now, the weight of the \(\mathcal{GWM}\) is computed as:

\[
    w(\mathcal{GWM}) = \sum_{e_G \in G} w(e_G) = \sum_{e_G \in G_1} w(e_G) + \sum_{e_G \in G_2} w(e_G) + \sum_{e_G \in G_3} w(e_G) + \sum_{e_G \in G} w(e_G)
\]
where \( e'_G \) are all leftover edges; they are due to the following reason: in \( G_3 \) when we add one \( e_G \) we add, at the same time, two \( e_M \) in \( M_3 \); then \( |G_3| \neq |M_3| \).

It is possible relate the weight of \( GWM \) and \( MWM \) as:

\[
\sum_{e_G \in G} w(e_G) \neq \sum_{e_M \in M} w(e_M)
\]

By looking at the subset of \( G \) and \( M \):

\[
\sum_{e_G \in G_1} w(e_G) = \sum_{e_M \in M_1} w(e_M)
\]

\[
\sum_{e_G \in G_2} w(e_G) \geq \sum_{e_M \in M_2} w(e_M)
\]

\[
\sum_{e_G \in G_3} w(e_G) \geq \frac{1}{2} \sum_{e_M \in M_3} w(e_M)
\]

Therefore:

\[
w(GWM) \geq \sum_{e_M \in M_1} w(e_M) + \sum_{e_M \in M_2} w(e_M) + \frac{1}{2} \sum_{e_M \in M_3} w(e_M)
\]

Let:

\[X_M = \sum_{e_M \in M_1} w(e_M) + \sum_{e_M \in M_2} w(e_M) + \frac{1}{2} \sum_{e_M \in M_3} w(e_M)\]

Then:

\[X_M \geq \frac{1}{2} \sum_{e_M \in M_1} w(e_M) + \frac{1}{2} \sum_{e_M \in M_2} w(e_M) + \frac{1}{2} \sum_{e_M \in M_3} w(e_M)\]

But:

\[\frac{1}{2} \sum_{e_M \in M_1} w(e_M) + \frac{1}{2} \sum_{e_M \in M_2} w(e_M) + \frac{1}{2} \sum_{e_M \in M_3} w(e_M) = \frac{1}{2} MWM\]

Therefore:

\[w(GWM) \geq \frac{1}{2} MWM\]

**Theorem 1 (Relation \( GWM \) and \( MWM \))**

\[w(GWM) \geq \frac{1}{2} MWM\]

**Corollary 1 (Relation \( GSM \) and \( MSM \))**

\[w(GSM) \geq \frac{1}{2} MSM\]

where:
. \textit{GSM} is the Greedy Size Matching;

. \textit{MSM} is the Maximum Size Matching.

This result is obtained by using as weights 0 and 1.

\textbf{Example 2 (Implementation)} Given the following matching with the same graphical notation reported here.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{example_graph.png}
  \caption{Example graph for implementation}
  \label{fig:example_graph}
\end{figure}

Algorithms steps.

\textbf{Iteration 1} The edge B-E is considered because is the one with the maximum edge:

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{iteration1.png}
  \caption{Iteration 1: Considered edge B-E}
  \label{fig:iteration1}
\end{figure}

This is a class 3 operation, therefore:

\begin{align*}
  G_3 &= B-E \\
  M_3 &= A-E, B-F
\end{align*}

\textbf{Iteration 2} Now the resultant graph is:

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{iteration2.png}
  \caption{Iteration 2: Resultant graph}
  \label{fig:iteration2}
\end{figure}

The considered edge is A-D; it belong to category 2:
therefore:

\[ G_2 = A-D \]
\[ M_2 = C-D \]

**Stop** No more edges \( e_M \) to be analysed, thus C-F is a *leftover*. 