Stability of $\mathcal{MWM}$

Notation

- N: switch size;
- n: time slot;
- $i \in [1\,,\,N]$: input i;
- $j \in [1\,,\,N]$: output j;
- $x_{ij}(n)$: VoQ size $i \to j$ at time n; it represents the queue length.

Given $x_{ij}(n)$, it is possible to define the matrix:

$$X(n) = [x_{ij}(n)]$$

For what concern arrivals:

$$a_{ij}(n) = \begin{cases} 1 & \text{arrival at input } i \text{ destinate to output } j \text{ at time } n \\ 0 & \text{otherwise} \end{cases}$$

Due to IID Bernoulli hypothesis:

$$a_{ij}(n) = \begin{cases} 1 & \text{wth probability } \lambda_{ij} \\ 0 & \text{wth probability } (1-\lambda_{ij}) \end{cases}$$

The term λ_{ij} is the *arrival rate*, therefore:

$$\lambda_{ij} = \mathbb{E}\left[a_{ij}\right] = 1 \cdot \lambda_{ij} + 0 \cdot (1-\lambda_{ij}) \tag{1}$$

Also in this case it is possible to define the matrix:

$$\Lambda = [\lambda_{ij}(n)]$$

This is the *arrival rate matrix*. Rates must satisfy **admissibility** properties:

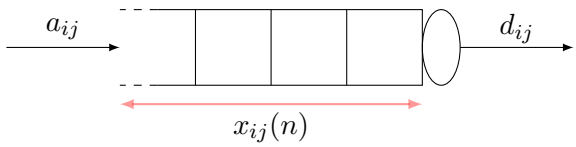
$$\begin{cases} \sum_i \lambda_{ij}(n) < 1 & \forall j \in [1\,,\,N] \\ \sum_j \lambda_{ij}(n) < 1 & \forall i \in [1\,,\,N] \end{cases}$$

The relation can not admit the "equal" because in that case the queue size tends to infinite, so the system is unstable. Thanks to this proof, it is possible to show also G/G/1 stability (because it is an 1×1 switch).

The service time is defined as:

$$d_{ij}(n) = \begin{cases} 1 & \text{if the VoQ}_{ij} \text{ is served} \\ 0 & \text{otherwise} \end{cases}$$

A graphical model is:



This model is not an M/D/1 queue: VoQs are not *work conserving*, so could happen that there are packets available, but not served by the specific VoQ$_{ij}$.

To model the matching concept through Matching (Permutation) Matrix, it has been used an $N \times N$ size matrix in which elements are binary. Example of a 3×3:

$$Matr_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad Matr_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

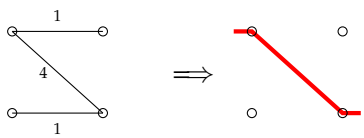
The first one maximizes the matching and just this type is accepted: globally, the number of maximum matching are $N!$ in an $N \times N$ system. The notation used is $M = [m_{ij}]$ such that:

$$\begin{cases} m_{ij} \in [0\,,\,1] \\ \sum_i m_{ij}(n) < 1 & \forall j \in [1\,,\,N] \\ \sum_j m_{ij}(n) < 1 & \forall i \in [1\,,\,N] \end{cases}$$

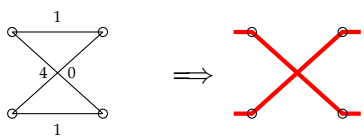
The service matrix is defined as:

$$D(n) = [d_{ij}(n)]$$

and it is, by construction, a matching matrix; it should deal just with complete matching:



is not valid while the following one yes:



The VoQ evolution is:

$$x_{ij}(n+1) = x_{ij}(n) + a_{ij}(n) - d_{ij}(n) \tag{2}$$

The weight of a matching is characterized by:

$$w = \sum_{i\,,\,j} x_{ij} \cdot m_{ij}$$

by knowing that:

$$X = [x_{ij}] \qquad M = [m_{ij}]$$

We assume $D[d_{ij}]$ be the Maximum Matching:

$$\mathcal{MWM} = \sum_{i\,,\,j} x_{ij} \cdot d_{ij}$$

Bounds:

. one possible lower bound is any kind of matching:

$$\sum_{i\,,\,j} x_{ij} \cdot m_{ij}$$

. one possible upper bound is:

$$\sum_{i\,,\,j} x_{ij}$$

. instead:

$$\frac{\sum_{i\,,\,j} x_{ij}}{N} \tag{3}$$

is a lower bound.

One possible example in which the $\mathcal{MWM}$ is equal to lower bound of 3 is:

$$X = [x_{ij}] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus we obtain:

$$\tilde{X} \cdot D = 2 \qquad \tilde{X}/N = 2$$

where

$$\tilde{X} = \sum_{i\,,\,j} x_{ij}$$

Theorems and definitions

Definition 1 Let M^κ be a matching matrix corresponding to the κ-th diagonal, where $\kappa \in [1\,,\,N]$. Examples in 3×3 environment:

$$M^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad M^3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We compare the $\mathcal{MWM}$ and its lower bound composed by m_{ij}^κ:

$$\begin{cases} \sum_{i\,,\,j} x_{ij} \cdot d_{ij} \geq \sum_{i\,,\,j} x_{ij} \cdot m_{ij}^1 \\ \sum_{i\,,\,j} x_{ij} \cdot d_{ij} \geq \sum_{i\,,\,j} x_{ij} \cdot m_{ij}^2 \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ \sum_{i\,,\,j} x_{ij} \cdot d_{ij} \geq \sum_{i\,,\,j} x_{ij} \cdot m_{ij}^N \end{cases}$$

By summing the two parts:

$$\sum_{\kappa=1}^{N} \sum_{i\,,\,j} x_{ij} \cdot d_{ij} \geq \sum_{\kappa=1}^{N} \sum_{i\,,\,j} x_{ij} \cdot m_{ij}^\kappa$$

By swapping in the second term the two sums:

$$N \cdot \sum_{i\,,\,j} x_{ij} \cdot d_{ij} \geq \sum_{i\,,\,j} \sum_{\kappa=1}^{N} x_{ij} \cdot m_{ij}^\kappa$$

Fixing x_{ij}, the sum over $\kappa \in [1\,,\,N]$ of m_{ij}^κ is simply the identity matrix, therefore:

$$N \cdot \sum_{i\,,\,j} x_{ij} \cdot d_{ij} \geq \sum_{i\,,\,j} x_{ij}$$

Thus:

$$\sum_{i\,,\,j} x_{ij} \cdot d_{ij} \geq \frac{\sum_{i\,,\,j} x_{ij}}{N}$$

Theorem 1 (Birkoff von Neumann) Let $A = [a_{ij}]$ be a matrix $N \times N$ where $a_{ij} \geq 0$ and $a_{ij} \in \mathbb{R}^+$. Moreover:

$$\sum_i a_{ij} \leq b \qquad \sum_j a_{ij} \leq b$$

Then, it is possible to write A as combination of matching matrices:

$$A \leq \sum_\kappa \alpha_\kappa \cdot M^\kappa$$

where:

. M^{κ} is a matching matrix;

. α_{κ} represents a *weight* and:

$$\sum_{\kappa} \alpha_{\kappa} = b$$

Example with:

$$b = 0.5 \qquad A = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \end{pmatrix}$$

Then:

$$A \leq 0.1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0.2 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 0.2 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Definition 2 $\Lambda(n) = [\lambda_{ij}(n)]$ is *admissible* if $\exists\, \varepsilon > 0$ such that:

$$\begin{cases} \sum_i \lambda_{ij}(n) < 1 - \varepsilon & \forall j \in [1\,,\, N] \\ \sum_j \lambda_{ij}(n) < 1 - \varepsilon & \forall i \in [1\,,\, N] \end{cases}$$

Definition 3 An IQ switch is *stable* if:

$$\lim_{n \to \infty} \left(\sup \sum_{i\,,\,j} x_{ij}(n) \right) < +\infty$$

where $\sum_{i\,,\,j} x_{ij}(n)$ is the total number of packets in VoQs at time n.

Theorem 2 (Lyapunov stability) Consider a VoQ characterized by $X(n)$ and let:

$$y = \sum_{i\,,\,j} x_{ij}(n) \tag{4}$$

Then, the IQ switch is stable if, for *large enough* $X(n)$ $\exists\, \varepsilon$ such that:

$$\lim_{y \to \infty} \frac{\mathbb{E}\left[\sum_{i\,,\,j} x_{ij}^2(n+1) - \sum_{i\,,\,j} x_{ij}^2(n) \,|\, X(n) \right]}{y} < -\varepsilon \tag{5}$$

This implies that, when there are lots of packets in the network at a given time slot, on average in the next slot time the queue size will decrease proportionally to $(\varepsilon \cdot y)$.

Stability proof

Theorem 3 The $\mathcal{MWM}$ is stable for any IID Bernoulli admissible traffic.

To prove the theorem we need to evaluate the numerator of equation 5. Since $X(n)$ is given, there is no need to specify $\mid X(n)$ and we indicate the numerator as:

$$\phi(X(n)) = \mathbb{E}\left[\sum_{i\,,j} x_{ij}^2(n+1) - \sum_{i\,,j} x_{ij}^2(n)\right]$$

The following simplification allows to obtain:

$$\phi(X(n)) = \mathbb{E}\left[\sum_{i\,,j} x_{ij}^2(n+1) - x_{ij}^2(n)\right] \tag{6}$$

Due to equation 2:

$$x_{ij}^2(n+1) = x_{ij}^2(n) + [a_{ij}(n) - d_{ij}(n)]^2 + 2x_{ij}(n)\cdot[a_{ij}(n) - d_{ij}(n)]$$

Now, the term $x_{ij}^2(n+1) - x_{ij}^2(n)$ in equation 6 becomes:

$$x_{ij}^2(n+1) - x_{ij}^2(n) = [a_{ij}(n) - d_{ij}(n)]^2 + 2x_{ij}(n)\cdot[a_{ij}(n) - d_{ij}(n)]$$

Therefore:

$$\phi(X(n)) = \mathbb{E}\left[\sum_{i\,,j} [a_{ij}(n) - d_{ij}(n)]^2 + 2x_{ij}(n)\cdot[a_{ij}(n) - d_{ij}(n)]\right]$$

By swapping the sum with the average:

$$\phi(X(n)) = \sum_{i\,,j} \mathbb{E}\left[[a_{ij}(n) - d_{ij}(n)]^2 + 2x_{ij}(n)\cdot[a_{ij}(n) - d_{ij}(n)]\right]$$

Since the dependence is just on the time slot n, it is possible remove it:

$$\phi(X) = \sum_{i\,,j} \mathbb{E}\left[[a_{ij} - d_{ij}]^2 + 2x_{ij}\cdot[a_{ij} - d_{ij}]\right] \tag{7}$$

The first term $\mathbb{E}\left[[a_{ij} - d_{ij}]^2\right]$ is bounded also when the queue size tends to ∞ because:

$$[a_{ij} - d_{ij}]^2 \leq 1 \implies 0.5\mathbb{E}\left[[a_{ij} - d_{ij}]^2\right] \leq 1 \implies 0.5\sum_{i\,,j} \mathbb{E}\left[[a_{ij} - d_{ij}]^2\right] \leq N^2$$

Thus:

$$\text{for } y \to \infty \implies 0.5\frac{N^2}{y} \text{ is negligible.}$$

Now it is possible to compute:

$$\lim_{y \to \infty} \frac{\phi(X)}{y} \tag{8}$$

where, the actual $\phi(X)$ is not the one reported in equation 7, but:

$$\phi(X) = \sum_{i\,,j} \mathbb{E}\left[2x_{ij} \cdot (a_{ij} - d_{ij})\right]$$

due to the simplification done. The only random variable in the last equation is just a_{ij} because d_{ij} is the maximum matching decided on the base of x_{ij}. Therefore:

$$\sum_{i\,,j} \mathbb{E}\left[2x_{ij} \cdot (a_{ij} - d_{ij})\right] = \sum_{i\,,j} 2x_{ij} \cdot (\mathbb{E}\left[a_{ij}\right] - d_{ij})$$

Since 1 holds :

$$\phi(X) = \sum_{i\,,j} 2x_{ij} \cdot (\lambda_{ij} - d_{ij}) \tag{9}$$

To describe if the traffic Λ is admissible we have exploited theorem 1:

$$\Lambda \leq \sum_{\kappa} \alpha_\kappa \cdot M^\kappa \quad , \quad \sum_{\kappa} \alpha_\kappa \leq 1 - \varepsilon$$

Thus:

$$\lambda_{ij} \leq \sum_{\kappa} \alpha_\kappa \cdot m_{ij}^\kappa$$

Therefore:

$$\sum_{i\,,j} x_{ij} \cdot \lambda_{ij} \leq \sum_{i\,,j} x_{ij} \cdot \sum_{\kappa} \alpha_\kappa \cdot m_{ij}^\kappa$$

By swapping the sums in the second term:

$$\sum_{i\,,j} x_{ij} \cdot \lambda_{ij} \leq \sum_{\kappa} \alpha_\kappa \cdot \sum_{i\,,j} x_{ij} \cdot m_{ij}^\kappa$$

We know that, for the second term, a possible bound is:

$$\sum_{\kappa} \alpha_\kappa \cdot \sum_{i\,,j} x_{ij} \cdot m_{ij}^\kappa \leq \sum_{\kappa} \alpha_\kappa \cdot \sum_{i\,,j} x_{ij} \cdot d_{ij}$$

Since:

$$\sum_{\kappa} \alpha_\kappa \leq 1 - \varepsilon$$

we obtain:

$$\sum_{\kappa} \alpha_\kappa \cdot \sum_{i\,,j} x_{ij} \cdot m_{ij}^\kappa \leq (1 - \varepsilon) \cdot \sum_{i\,,j} x_{ij} \cdot d_{ij}$$

Now it is possible say that 9 has the following bound:

$$\sum_{i,j} 2x_{ij} \cdot (\lambda_{ij} - d_{ij}) \leq 2 \cdot \left\{ \left[(1-\varepsilon) \cdot \sum_{i,j} x_{ij} \cdot d_{ij} \right] - \left[\sum_{i,j} x_{ij} \cdot d_{ij} \right] \right\}$$

It becomes:

$$\sum_{i,j} 2x_{ij} \cdot (\lambda_{ij} - d_{ij}) \leq -2\varepsilon \cdot \sum_{i,j} x_{ij} \cdot d_{ij}$$

Due to equation 3, the second term can be rewritten obtaining:

$$\sum_{i,j} 2x_{ij} \cdot (\lambda_{ij} - d_{ij}) \leq -2\varepsilon \cdot \frac{\sum_{i,j} x_{ij}}{N}$$

So:

$$\phi(X) \leq -2\varepsilon \cdot \frac{\sum_{i,j} x_{ij}}{N}$$

According to equation 8:

$$\lim_{y \to \infty} \frac{\phi(X)}{y} \leq \frac{-2\varepsilon \cdot \frac{\sum_{i,j} x_{ij}}{N}}{y}$$

Recalling equation 5:

$$\lim_{\sum_{i,j} x_{ij} \to \infty} \frac{\phi(X)}{\sum_{i,j} x_{ij}} \leq \frac{-2\varepsilon \cdot \frac{\sum_{i,j} x_{ij}}{N}}{\sum_{i,j} x_{ij}} \implies 0.5 \lim_{\sum_{i,j} x_{ij} \to \infty} \frac{\phi(X)}{\sum_{i,j} x_{ij}} \leq \frac{-2\varepsilon}{N}$$

In a more compact form:

$$\lim_{y \to \infty} \frac{\phi(X)}{y} \leq -2 \cdot \frac{\varepsilon}{N}$$